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THE CALCULUS OF INDIVIDUALS AND ITS USES[1]

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I. An individual or whole we understand to be whatever is represented in any given discourse by signs belonging to the lowest logical type of which that discourse makes use. What is conceived as an individual and what as a class is thus relative to the discourse within which the conception occurs. One task of applied logic is to determine which entities are to be construed as individuals and which as classes when the purpose is the development of a comprehensive systematic discourse.

The concept of an individual and that of a class may be regarded as different devices for distinguishing one segment of the total universe from all that remains. In both cases, the differentiated segment is potentially divisible, and may even be physically discontinuous. The difference in the concepts lies in this: that to conceive a segment as a whole or individual offers no suggestion as to what these subdivisions, if any, must be, whereas to conceive a segment as a class imposes a definite scheme of subdivision—into subclasses and members.[2]

The relations of segments of the universe are treated in traditional logistic at two places, first in its theorems concerning the identity and diversity of individuals, and second in its calculus of membership and class-inclusion. But further relations of segments and of classes frequently demand consideration. For example, what is the relation of the class of windows to the class of buildings? No member of either class is a member of the other, nor are any of the segments isolated by the one concept identical with segments isolated by the other. Yet the classes themselves have a very definite relation in that each window is a part of some building. We cannot express this fact in the language of a logistic which lacks a part-whole relation between individuals unless, by making use of some special physical theory, we raise the logical type of each window and each building to the level of a class—say a class of atoms—such that any class of atoms that is a window will be included (class-inclusion) in some class that is a building. Such an unfortunate dependence of logical formulation upon the discovery and adoption of a special physical theory, or even upon the presumption that such a suitable theory could in every case be discovered in the course of time, indicates serious deficiencies in the ordinary logistic. Furthermore, a raising of type like that illustrated above is often precluded in a constructional system by other considerations governing the choice of primitive ideas.

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[1] A somewhat elaborated version of a paper read in Cambridge, Mass., before a joint meeting of the Association for Symbolic Logic and the American Philosophical Association, Eastern Division, on December 28, 1936.

[2] The relation is somewhat analogous to the more familiar one between classial and serial concepts, dealing as they do with the same material, but in a manner that makes the latter more highly specialized.

The ordinary logistic defines no relations between individuals except identity and diversity. A calculus of individuals that introduces other relations, such as the part-whole relation, would obviously be very convenient;[3] but what chiefly concerns us in this paper is the general applicability of such a calculus to the solution of certain logico-philosophical problems.

The calculus of individuals we shall employ is formally indistinguishable from the general theory of manifolds developed by Leśniewski.[4] Leśniewski's purpose, quite different from ours, was to establish a general theory of manifolds that would not be subject to Russell's paradox; but since he excludes the notion of a null class, his formal system is virtually the same as that which we interpret as a calculus of individuals. Inasmuch as his system is rather inaccessible, lacks many useful definitions, and is set forth in the language of an unfamiliar logical doctrine and in words rather than symbols, we shall attempt (in Part II) to restate the calculus in more useable form, with additional definitions, a practical notation and a transparent English terminology. In Part III we shall explain how this calculus enables us to describe generally certain important, but often neglected properties of relations, and thereby contributes to the clarification of many philosophical problems.

II. The general features of the abstract calculus may perhaps be most readily apprehended by comparison with the Boolean algebra of classes. It involves operations of addition, multiplication, and negation, a part-whole relation analogous to class-inclusion and an element analogous to the Boolean universal class. It differs from the Boolean analogue in ways consequent upon the refusal to postulate a null element, although the primitive relation of "discreteness" may be correlated with the Boolean function "$x \cdot y = 0$".

In the light of this analogy, the characteristic propositions of the calculus may be generally described: To any analytic proposition of the Boolean algebra will correspond a postulate or theorem of this calculus provided that, when in the Boolean proposition every expression of the form "$x \cdot y = 0$" is replaced by an expression of the form "x is discrete from y", no reference to the null element remains and every product and negation is either deducibly unequal to the null element or else is conditionally affirmed to be unequal to it. From the three postulates (presented below) of the formal calculus, enough theorems have been deduced to indicate that this characterization is accurate. Some illustrative theorems appear in the sequel.

For the formal establishment of the calculus, the symbolism and logistic of Whitehead and Russell's *Principia mathematica* have been employed in order to secure correlation with other logical doctrines. Only the one primitive idea already mentioned is required: the dyadic propositional function, or relation, written "$x \rceil\!\lfloor y$" and here interpreted to mean that the individuals which are its arguments have no part in common, that they are discrete.[5] In our interpretation, furthermore, parts and common parts need not necessarily be spatial parts. Thus in our applications of the calculus to philosophic problems, two concrete entities, to be taken as discrete, have not only to be spatially discrete, but also temporally discrete, discrete in color, etc., etc.

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[3] Since this paper was presented, the convenience of such a calculus of individuals has been well illustrated by Dr. J. H. Woodger's ***Axiomatic method in biology*** (1937).

[4] In *O podstawach matematyki* (in Polish), ***Przegląd filozoficzny,*** vols. 30–34 (1927–31).

[5] Leśniewski employs discreteness as his primitive relation in the final version of his system. See Chapter X of his above-mentioned paper.

In terms of the one primitive idea just described, other concepts may be defined as follows:

I.01 $$x < y =_{\mathrm{Df}} . z \rceil\!\!\lfloor y \supset_z z \rceil\!\!\lfloor x$$

I.e., one thing is a part of another if whatever is discrete from the latter is also discrete from the former.

I.011 $$x \ll y =_{\mathrm{Df}} x < y \,.\, x \neq y$$[6]

This defines *proper part* in a familiar sense.

I.02 $$x \circ y =_{\mathrm{Df}} (\exists z) \,.\, z < x \,.\, z < y$$

I.e., two things overlap if they have a part in common. As postulate I.13 (below) indicates, overlapping is equivalent to the denial of the primitive discreteness.

I.03 $$x \,\mathrm{Fu}\, \alpha =_{\mathrm{Df}} z \rceil\!\!\lfloor x \,.\, \equiv_z . \, y \in \alpha \supset_y z \rceil\!\!\lfloor y$$

This defines the heterogeneous relation of an individual, x, which is the fusion, or sum-individual, of a class, α, to that class. An individual is said to stand in that relation to a class when everything that is discrete from it is discrete from every member of the class and everything discrete from every member of the class is discrete from it. Postulate I.1 affirms that any not null class has a sum, and it is demonstrated in subsequent theorems that the relation here defined is a one-many relation, so that we may speak of *the sum*, or fusion, of any existent class, α.

I.04 $$x \,\mathrm{Nu}\, \alpha =_{\mathrm{Df}} z < x \,.\, \equiv_z . \, y \in \alpha \supset_y z < y$$

This defines the relation of an individual which is the nucleus, or product-individual, of a class, to that class. It is structurally similar to definition I.03, with the part-whole relation replacing the discreteness relation. Theorem I.56 and subsequent theorems reveal that for any not-null class of individuals such that some one individual is a common part of every member of the class, there is at least one individual which is a nucleus of that class; and that no class has more than one nucleus, so that we may speak of *the* nucleus of the class.

The concepts of fusion and nucleus just defined are not strictly Boolean concepts, since the Boolean concepts operate within one logical type, whereas

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[6] Identity is not defined nor taken as primitive since it is already defined in our logistic vehicle, *Principia mathematica*. Had it been desirable to develop this calculus in isolation from other treatments of logistic, identity could have been defined as mutual part-whole, to which it is equivalent by theorem I.315:

$$x = y \,.\, \equiv . \, x < y \,.\, y < x$$

This isolation, however, seemed undesirable because of the uses, as in Part III below, that we intended to make of the calculus.

these relations are heterogeneous, relating concepts of one type with those of the next higher type. They correspond, however, to the sums and products of classes defined in *Principia mathematica*, *40.01 and *40.02. But whereas sums and products defined in *Principia mathematica* are only applicable to classes of classes, the notions here defined are precisely applicable to classes of individuals. It is, furthermore, by means of these non-Boolean relations that analogues to the strictly Boolean operations of addition and multiplication are defined in I.06 and I.07 (below).

Definition I.05 introduces *the Universe*, or the universal element, U, and I.08 defines the operation of negation:

I.05 $\mathrm{U} =_{\mathrm{Df}} \mathrm{Fu}\text{‘}\mathrm{V}$

I.06 $x + y =_{\mathrm{Df}} \mathrm{Fu}\text{‘}(\iota x \cup \iota y)$

I.07 $xy =_{\mathrm{Df}} \mathrm{Nu}\text{‘}(\iota x \cup \iota y)$

I.08 $-x =_{\mathrm{Df}} \mathrm{Fu}\text{‘}\hat{y}(y \rceil\!\lfloor x)$[7]

Through the introduction of the non-Boolean fusion and nucleus, propositions having no Boolean analogues appear in our calculus. Such propositions state more general laws, of an obvious kind, governing the Boolean operations, and from them the analogues to the Boolean propositions are, in fact, deduced as more or less direct corollaries. Yet although these general relations of individuals to classes have here been used to define the Boolean operations and the universal element, independent definitions could have been given:

I.05′ $\mathrm{U} =_{\mathrm{Df}} (\text{℩}x)((y) \,.\, y<x)$

I.06′ $x + y =_{\mathrm{Df}} (\text{℩}z)(w \rceil\!\lfloor z \,.\, \equiv_w .\, w \rceil\!\lfloor x \,.\, w \rceil\!\lfloor y)$

I.07′ $xy =_{\mathrm{Df}} (\text{℩}z)(w < z \,.\, \equiv_w .\, w < x \,.\, w < y)$

I.08′ $-x =_{\mathrm{Df}} (\text{℩}z)(z \rceil\!\lfloor x \,.\, x + z = \mathrm{U})$

Replacement of I.05 to I.08 by these alternative definitions would, of course, necessitate alterations in the postulates. It would also result in the elimination from the calculus of all the highly important non-Boolean propositions.

Of the three postulates, the first allows us to assert the existence of some individual which is the fusion of a given class whenever that class is not null; the second relates the discreteness calculus to identity, already defined in *Principia mathematica*; while the third in effect states a general property of the primitive relation:

I.1 $(\exists x) \,.\, x \in \alpha \,.\, \supset .\, (\exists y) \,.\, y \,\mathrm{Fu}\, \alpha$[8]

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[7] The present form of this definition has been suggested by the corresponding definition in Dr. A. Tarski's appendix to Woodger, op. cit.

[8] Leśniewski employs only two postulates. One is identical with our I.13 expanded in terms of the primitive relation; the other asserts both the existence and uniqueness of the fusion of any (non-null) class. Our postulate I.1 is weaker, since it asserts only the existence of some such individual but not its uniqueness. Accordingly we require also postulate I.12.

I.12 $x < y \,.\, y < x \,.\supset.\, x = y$

I.13 $x \mathbin{\boldsymbol{\circ}} y \equiv \sim(x \mathbin{\rceil\!\!\lfloor} y)$

Some illustrative theorems are:

I.3 $x < y \,.\, y < z \,.\supset.\, x < z$

I.31 $x < x$

I.e., "part-whole" is a transitive and reflexive relation. It is non-symmetrical.

I.325 $\sim(x \ll x)$

I.326 $x \ll y \supset \sim(y \ll x)$

I.328 $x \ll y \,.\, y \ll z \,.\supset.\, x \ll z$

I.e., "proper part" is an irreflexive, asymmetrical, but transitive relation.

I.327 $x \ll y \,.\, y < z \,.\supset.\, x \ll z$

Theorems like I.327 may be proven for every possible permutation of "part" and "proper part" except the one "$x<y.y<z.\supset.x\ll z$."

I.331 $x \mathbin{\boldsymbol{\circ}} y \equiv y \mathbin{\boldsymbol{\circ}} x$

I.332 $x < y \supset x \mathbin{\boldsymbol{\circ}} y$

I.333 $x \mathbin{\boldsymbol{\circ}} x$

I.53 $(\exists x) \,.\, x \in \alpha \,.\equiv.\, E!\,\mathrm{Fu}\text{‘}\alpha$

I.55 $E!\,\mathrm{Fu}\text{‘}\alpha \,.\supset.\, \alpha \subset \beta \supset \mathrm{Fu}\text{‘}\alpha < \mathrm{Fu}\text{‘}\beta$

I.556 $E!\,\mathrm{Fu}\text{‘}\alpha \,.\supset.\, \alpha = \beta \supset \mathrm{Fu}\text{‘}\alpha = \mathrm{Fu}\text{‘}\beta$

Propositions asserting conditionally the converse implications of those in the consequents of I.55.556 are not theorems of the system. Many distinct classes may have the same fusion. For example, let α be the class of tables, β be the class of table-tops, and γ be the class of frames (including legs, drawers etc.) of tables. It is plain that

$$\alpha \cap (\beta \cup \gamma) = \Lambda$$

That is, α and $\beta \cup \gamma$ are distinct classes and no member of one is a member of the other. Yet they isolate the same part of the total universe:

$$\mathrm{Fu}\text{‘}\alpha = \mathrm{Fu}\text{‘}(\beta \cup \gamma)$$

They differ in the manner of subdivision that they prescribe for that part.

Sample theorems concerning the existence and uniqueness of the nucleus of a class and the negate of an individual are:

I.56 $(p\text{‘}\overrightarrow{<}\text{“}\alpha) \neq \Lambda \,.\supset (\exists x) \,.\, x \,\mathrm{Nu}\, \alpha$

I.57 $x \,\mathrm{Nu}\, \alpha \,.\, z \,\mathrm{Nu}\, \alpha \,.\supset.\, x = z$

I.58 $(p\text{‘}\,\overrightarrow{<}\,\text{“}\alpha) \neq \Lambda \,.\, \supset \mathrm{E!}\,\mathrm{Nu}\text{‘}\alpha$

I.59 $(\exists y) \,.\, y \rceil\!\!\lfloor x \,.\, \equiv \,.\, \mathrm{E!}\,\text{-}x$

As special cases of general theorems concerning fusions we have:

I.6 $\mathrm{E!}\, x + y$

I.62 $x + y = y + x$

I.66 $(x + y) + z = x + (y + z)$

Theorems analogous to the above may be demonstrated for "xy", but only conditionally upon the assumption that "$x \circ y$", or, what is equivalent, "$\mathrm{E!}xy$". Analogues to DeMorgan's formula appear under interesting conditions. For example,

I.85 $x \circ y \,.\, x \neq \mathrm{U} \,.\, y \neq \mathrm{U} \,.\, \supset \,.\, xy = \text{-}(\text{-}x + \text{-}y)$

III. Besides supplementing the body of symbolic logic, the calculus of concepts of lowest type equips us to exhibit and deal efficaciously with certain relational properties which are often ignored or misunderstood, sometimes to the detriment of constructional undertakings like Carnap's *Logischer Aufbau der Welt.*[9]

Consider, for example, the relation "met with"; we cannot define the ordinary meaning of propositions to the effect that three or more people all met together by requiring that every pair met; for each pair may have met separately, without all three ever having met together. Or again, John and James may be lodge-brothers, and James and Arthur be lodge-brothers, and John and Arthur be lodge-brothers, without all three being brothers in any one lodge.[10] Even the commonplace proposition that a color, C, is, as we customarily say, "at place P at time T" is not implied by the proposition that C is at place P and at time T.

In each of these cases we encounter a relation which is significant in varying degrees; as dyadic, triadic, tetradic, and so on. The relation "met with", for example, may obtain between two people, among three, among four, and indeed among any number. Such a relation without any fixed degree may be called a "multigrade" relation.[11]

Customary methods of treating the logic of relations will admit of no simple introduction of multigrade relations, for these methods presume that an exhaustive and exclusive classification of relations can be set up in terms of their definite degree. Two alternative interpretations will, however, provide for their introduction in a manner consistent with this presumption. Either they may be construed as each a series of relations, the successive members of which have successively higher degrees, or they may be construed as predicates taking classes of various magnitudes for their arguments.

The second interpretation is strictly applicable only in the case of a multigrade relation that is thoroughly symmetrical. In other cases, it must be modified in the manner indicated below (p. 53). Since, however, the relations that we shall first consider are all of them thoroughly symmetrical, we adopt this interpretation in the sequel.

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[9] Berlin, 1928. See especially sections 67–93, pp. 108–120. This work is subsequently referred to as the *Aufbau.*

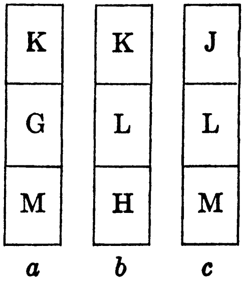
[10] The following discussion of the problem here illustrated is that referred to by W. V. Quine in *Relations and reason*, ***Technology review***, vol. 41 (1939), pp. 325, 327n.

[11] A unigrade relation is a relation of any one degree; a multigrade relation is one having at least two different degrees.

But whichever view of multigrade relations is preferred, the notable feature of the examples previously given, although it does not attach to all multigrade relations by any means, is that a given degree of the relation or predicate cannot be reduced, by the use of none but the commonly recognised logical devices, to its lower degrees. It is where this reduction is impossible that a crucial problem confronts the constructionalist; for considerations of economy, together with the special conditions of his problem in hand, will ordinarily cause the constructionalist to reject as primitives of his system any predicate taking anything other other than individuals as arguments, any whole hierarchy of relations, and any uppermost member of such a hierarchy (should it be finite) when the identification of that upper limit requires postponement of the formal development pending the detailed investigation of contingent matters of fact. Thus he is confronted with a dilemma: if multigrade relations be admitted without interpretation, then the standard logic of relations, developed in terms of a classification of relations according to degree, is inapplicable to them, but if they are admitted by using either interpretation suggested above, then often they are not themselves and cannot be reduced to acceptable primitives.

The manner in which the calculus of individuals may be enlisted to simplify the primitives needed, to display the connection between the different degrees of the relation or predicate, and to express the distinction in meaning between saying, for example, that three men all met together and saying that each two met severally, can best be illustrated by a slightly more complicated example than any of those yet given.

Suppose we have as elements a set of three columns, each colored with three bands, as pictured in the accompanying diagram, and suppose that the relation S is such that "xSy" means that in some one band—lower, middle, or upper—the two entities x and y are identically colored (in the sense that no color in that band either in x or in y is different from any one color in that band in the other). It is clear that S is a relation like those already considered; that we may have three columns, like the ones pictured, such that aSb, bSc, and aSc, even though all three columns have no single color in any one band.



The capital letters represent
distinct shades of color

However, nothing in our specification of S prevents it from taking as relata, not only single columns, or elements, but also those entities that are sums of the elements; the expression "$xSy+z$" has the perfectly clear and unambiguous meaning that x and $y+z$ are identically colored, in the sense described, at some level. In the illustration given, therefore, "$aSb+c$" is false, for at whatever level we look, either a and $b+c$ have entirely different colors, or else a is unicolored while $b+c$ is bi-colored, and the condition for the holding of S is not satisfied in either case. The proposition "$xSy+z$" will thus hold only if x and $y+z$—and, therefore, x, y and z—have a single, identical color at some one level; and accordingly the triadic degree of the relation may be defined by the function "$xSy+z$." The principle involved is generally extensible to any degree; hence we may define the class-predicate, S', applicable to classes having any finite number of members, as follows:

$$S'(\alpha) =_{\mathrm{Df}} \beta \neq \Lambda \,.\, \gamma \neq \Lambda \,.\, \beta \cap \gamma = \Lambda \,.\, \beta \cup \gamma \subset \alpha \,.\, \supset_{\beta,\gamma} .\ \mathrm{Fu}`\beta\ S\ \mathrm{Fu}`\gamma$$

That is, "$S'(\alpha)$" means that for every two discrete subclasses, β and γ, of α, the fusion of β has the relation S to the fusion of γ.

Because the particular relation, S, chosen happens to have a certain property later to be defined and called "interdissectiveness," the expressions "$xSy+z$," "$ySx+z$," and "$zSx+y$" are all equivalent, and are equivalent to "$S'(\{x,y,z\})$." Taking advantage of this property, we can simplify our general definition to read as follows:

$$S'(\alpha) =_{\mathrm{Df}} (\exists \beta,\gamma) : \beta \cap \gamma = \Lambda : \beta \cup \gamma = \alpha \,.\, \mathrm{Fu}`\beta\ S\ \mathrm{Fu}`\gamma$$

That is, "$S'(\alpha)$" means that the relation S holds between the fusions of some two mutually exclusive and exhaustive subclasses of α.[12]

The meaning, then, of saying that x has the relation S to y and z together, not merely severally, is given by the expression "$xSy+z$"; and the same treatment is applicable to the other relations that we noted earlier. That a man Smith met with Jones and Brown together means that he met with the entity which is the sum of the two. The sum will not be a person, of course, but is a definable though discontinuous whole. For a color to be "at a place at a time" is for it to be at that entity which is comprised of both, the place plus the time, or in other words, at the place-time. Likewise for a man to be a lodge-brother of two others together is for him to be a lodge-brother of their sum.

But caution is necessary here: the suggested treatment may be employed only when the relation in question is one, like our S, that takes as relata not merely atomic elements, but also sums of these elements, and the interpretation of the relation must be constant irrespective of the particular relata satisfying it in a particular case. If, for instance, "lodge-brother" be understood as a relation holding solely among individual *persons,* we could not speak of a man being the lodge-brother of the sum of two other persons. We should then have to employ as our primitive a broader relation, say "lodge-affiliate," which is not thus restricted and in terms of which the narrower relation could be defined.

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[12] The two definitions just proposed are equivalent only for cases in which the cardinality of the argument, α, is greater than or equal to two. The first definition makes "$S'(\alpha)$" true when the cardinality of α is less than two, while the second makes it false under the same circumstances. This difference, however, rests upon trivial cases that may be decided by considerations of convenience in dealing with the particular subject-matter and problem in hand; and either definition may be easily adjusted to accord with whatever decision is made.

The general nature of the technique employed in the preceding analyses might be described as follows.

Given for analysis a multigrade relation, its dyadic cases are abstracted and taken as a primitive dyadic relation. The multigrade relation itself, represented in the illustration just given by S', is then definitionally introduced as the predicate of a class. No restriction is put upon the cardinality of the argument-classes that satisfy it. The definiens of the multigrade predicate involves references to the primitive dyadic case of the relation in question. But many of the arguments assigned in the definiens to this dyadic case are describable only by the use of concepts developed in the calculus of individuals; generally they are sums of individuals or fusions of classes of individuals.

(The general treatment just described would have to be slightly modified in the case of a multigrade relation that is not wholly symmetrical. "To murder" and "to annoy" are multigrade, inasmuch as one person may murder or annoy another, or several persons may cooperate together in these actions; but not wholly symmetrical, inasmuch as interchanging terms designating the murdered man and one of the accomplices will not generally result in a proposition equivalent to the original. All multigrade relations are, however, at least partially symmetrical. For example, arguments designating the accomplices in a murder may be interchanged at will without affecting the truth-value of the propositions under consideration. Such multigrade relations may be definitionally introduced as heterogeneous relations between the class of the interchangeable individuals and the others.)

That our general technique enables us to introduce multigrade relations in a manner conformable with the guiding principles of constructionalism already suggested (on page 51), is due to the fact that $a+b$ is a concept of the same logical type as a and b themselves, and more generally that the fusion of a class, α, is a concept of the same logical type as the members of α. Thus "$aRb+c$," and "$\mathrm{Fu}`\alpha R \mathrm{Fu}`\beta$" express dyadic relations of individuals; they are expressions of the form "xRy" in which the individuals concerned happen to be referred to by more explicit descriptions. Therefore our general definition, for example of the predicate S', has as its definiens a logical function of a strictly dyadic relation of individuals.

It may now be noticed that if the general technique just outlined had been used in defining a "quality-class" in the system of Carnap's *Aufbau,* the gravest defects of the early constructions would have been avoided. For the abstraction of these quality classes is supposed to be accomplished through the use of a relation of similarity that is, in all relevant respects, like the relation S of the preceding illustration. And the defects in the construction consist in, or result from, mistakenly supposing that a class of things each member of which is similar to each other is a class of things which are all similar.

What our procedure accomplishes is the analysis of any such complicated and easily mishandled special concept as S' into a much simpler special concept and the general relation of discreteness, which is apart from any particular system and is applicable wherever a problem of multigrade relations arises. The problems of the men who met together, of the lodge-brothers, and of the colored columns are met in each case by employing a special simple relation together with the general discreteness relation. To adopt a complicated predicate of classes as primitive in each case would be to forego an analysis which discloses the identity of the logical problem present in all these cases.

The details of the solutions of constructional problems of the general character of those illustrated will vary with the specific properties possessed by the relations in question, especially those properties that the calculus of individuals assists us to comprehend and define. The relations so far considered have all been non-agglomerative, that is, have not had the property defined in D5 below. This was the feature that made impossible the definition of the higher degrees of any of these relations in terms of the lower, so long as the atomic elements alone were considered. On the other hand, all these relations were also not merely dissective—conforming to D6—but were even interdissective—conforming to D7; and it was for this reason that "$aSb+c$," "$bSa+c$," and "$cSa+b$" were equivalent.

Many further properties of relations and predicates may be defined, and may prove important in connection with other problems. A few of the more interesting of these properties are defined as follows:

D1 R is *internal* if $x \; R \; y \supset_{x,y} x \circ y$

D2 R is *external* if $x \; R \; y \supset_{x,y} x \urcorner\!\llcorner \, y$

A good deal of the traditional discussion concerning internal and external relations might be clarified by consideration of these two simple, but systematic, definitions. The relation of a part to a whole and of a quality to the object that it qualifies are examples of internal relations. Things with no common properties and with divers locations in space and time are in general externally related.

D3 R is *expansive* if $x \; R \; y \supset_{x,y,z} x \; R \; y + z$

Part-whole and overlapping are expansive relations.

D4 R is *cumulative* if $x \; R \; y \, . \, x \; R \; z \, . \supset_{x,y,z} . \; x \; R \; y + z$

Any expansive relation is cumulative, but the converse is not always true. Discreteness, for example, and the converse of part-whole are cumulative without being expansive.

D5 R is *agglomerative* if $x \; R \; y \, . \, x \; R \; z \, . \, y \; R \; z \, . \supset_{x,y,z} . \; x \; R \; y + z$

Every cumulative—and hence also every expansive—relation is agglomerative, but the converse is not true. The relations of similarity and of meeting that we analysed in the earlier pages of this article are non-agglomerative, and hence non-cumulative and non-expansive. The converse of "proper part of"—defined above in I.011 (p. 47)—is agglomerative but not cumulative and hence not expansive.

D6 R is *dissective* if $x \ R \ y \ . \ z < y \ . \supset_{x,y,z} . \ x \ R \ z$

The converse of the part-whole relation is dissective. Also the relation S, analysed above, is dissective on the tacit assumption that the original elements—a, b, and c in the illustration—are to be construed as unanalysable into proper parts.

D7 R is *interdissective* if $x \ R \ y \ . \ w < x + y \ . \ z < x + y \ . \supset_{w,x,y,z} . \ w \ R \ z$

Every interdissective relation is dissective, but the converse is not always true. The relation S, analysed above, is interdissective. The converse of part-whole is dissective but not interdissective.

It would be possible to extend this list of definitions by defining all the variants of the above properties that could be distinguished through the possible non-symmetry of the relation involved. For example, R is *counterexpansive* if $xRy \supset_{x,y,z} x+zRy$. There are seventeen such variants of the property defined in D5. When the relation is symmetrical, the variants will be equivalent to the properties defined above. Also certain of these properties are only special applications to relations of properties that might be defined for predicates generally; for example, ϕ is *expansive* if $\phi(y) \supset_{y,z} \phi(y+z)$.

IV. The utility of the proposed calculus of individuals is by no means limited to its usefulness in treating the problems considered in this paper. Because it is simple in structure, because its primitive and defined ideas closely parallel intuitive notions, because its major non-symmetrical relation (namely, "part-whole") is transitive, because it introduces relationships other than membership (namely, "fusion of" and "nucleus of") between individuals and classes, and especially because it provides means for treating many varied entities by means of concepts of a single logical type, the calculus of individuals is a powerful and expedient instrument for constructional work. In addition it performs the important service of divorcing the *logical* concept of an individual from metaphysical and practical prejudices, thus revealing that the distinction and interrelation of classes and wholes is capable of a purely formal definition, and that both concepts, and indeed all the concepts of logic, are available as neutral tools for the constructional analysis of the world. Then, for example, it becomes clear that the practice of supposing that *things* are what the x's and y's of *Principia mathematica* denominate and that qualities are necessarily to be interpreted as logical predicates thereof, rather than vice versa, is purely a matter of habit. The dispute between nominalist and realist as to what actual entities are individuals and what are classes is recognized as devolving upon matters of interpretative convenience rather than upon metaphysical necessity.

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